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Steady fluid flow in a precessing spheroidal shell

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The linear boundary-layer analysis by Stewartson & Roberts (1963) and by Roberts & Stewartson (1965) for the motion of a viscous fluid inside the spheroidal cavity of a precessing rigid body is extended to include effects due to the non-linear terms in the boundary-layer equation. The most significant consequence is a differential rotation super-imposed on the constant vorticity flow given by the linear theory. In addition it is shown that a tidal bulge of the cavity forces a fluid motion similar to that caused by the precessional torque. The relevance of both effects for the liquid core of the earth is briefly discussed.

1. Introduction

Among the problems of fluid flow in rotating systems, the motion of a viscous fluid inside a precessing spheroidal shell has attracted special interest, not only because of its relevance to geophysical and astrophysical questions but also because of its unusual properties. Although the geometry of the problem is simple and only a few parameters enter into its physical description, it shows a number of interesting and often unexpected phenomena. Many of these, especially those with non-stationary and turbulent structure, can be investigated only experimentally at the present time. For the experimental investigation of the problem we refer to the forthcoming paper by Malkus (1968). In the present paper we are concerned with the theoretical description of the laminar flow, which is stationary in the frame of reference which rotates with the precession rate.

A detailed analysis of the mathematical problem has been given by Roberts & Stewartson (1963, 1965). In their second paper they derive an exact solution of the Navier–Stokes equation of motion under the assumption that the flow has constant vorticity. This solution does not, however, satisfy the non-slip condition at the rigid boundary. In a different approach Roberts & Stewartson use boundary-layer methods restricting the analysis to small amplitudes for which the boundary-layer equations can be linearized. In accordance with their first solution, they conclude that the interior flow has constant vorticity. It will be shown that this conclusion is not valid for finite amplitudes. A differential rotation of the interior fluid body is caused by the non-linear advection of the boundary-layer velocity field. Hence the exact solution of the equations of motion by Roberts & Stewartson is not approached in the limit of vanishing

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viscosity. For this reason the precessional problem, in which the direction of the rotation vector of the container varies, differs considerably from the so-called spin-up problem in which only the rate of rotation is changed. While in the latter case it has been concluded by Greenspan & Weinbaum (1965) that the finite-amplitude results differ only quantitatively from the results given by the linear theory, qualitatively new effects are introduced due to the non-linear terms in the precessional problem.

After formulating the problem and the method of solution in § 2 we shall recapitulate the results of the linear theory in §3. The differential rotation superposed on the constant vorticity flow given by the linear theory will be derived in §4. In §5 we shall show that the action of a tidal bulge of the shell is similar to that of the precessional force. A short discussion of the relevance of both effects for the earth's core is given in §6.

2. The mathematical formulation of the problem

We are considering the stationary flow of an incompressible fluid inside a spheroidal shell which is rotating about its axis with the constant angular velocity $\boldsymbol{\omega}_s$ relative to a rotating frame of reference. The constant angular velocity $\omega_s \boldsymbol{\Omega}$ of the frame of reference corresponds to the precession rate with which the axis of the shell precesses relative to the inertial space. We introduce the radius a of the spheroidal cavity perpendicular to its axis as length-scale and $\omega_s a$ as scale for the velocity. In this dimensionless description the angular velocity of the container is given by the unit vector $\mathbf{k}$, and $\boldsymbol{\Omega}$ is the angular velocity of the rotating frame of reference. The equations for the fluid velocity vector $\mathbf{q}$ are

$$\nabla . \mathbf{q} = 0,$$
$$2\boldsymbol{\Omega} \times \mathbf{q} + \mathbf{q} . \nabla \mathbf{q} = -\nabla p + E \nabla^2 \mathbf{q}. \tag{2.1}$$

The kinematic viscosity ν enters the equation in the definition of the Ekman number, $E = \nu/\omega_s a^2$.

In the following discussion of (2.1) together with the viscous boundary condition

$$\mathbf{q} = \mathbf{k} \times \mathbf{r} \quad \text{on } \Sigma, \tag{2.2}$$

we will assume that the viscosity is small, and hence neglect to the first order the viscous term outside a thin boundary layer close to the wall. According to this boundary-layer assumption, the velocity vector is given by

$$\mathbf{q} = \mathbf{q}_i + \tilde{\mathbf{q}}, \tag{2.3}$$

where $\mathbf{q}_i$ describes the velocity field throughout the interior while $\tilde{\mathbf{q}}$ is the additional boundary-layer velocity vector which is non-vanishing only in a thin layer close to the wall and exponentially decreasing toward the interior. For a general formulation of the boundary-layer method for the solution of the equations in a rotating system, we refer to Greenspan (1965). Using his notation we assume the following expansion for $\mathbf{q}_i$ and $\tilde{\mathbf{q}}$ in powers of $\sqrt{E}$:

$$\mathbf{q}_i = \mathbf{q}_0 + \sqrt{E}\, \mathbf{q}_1 + \dots, \tag{2.4}$$
$$\tilde{\mathbf{q}} = \tilde{\mathbf{q}}_0 + \sqrt{E}\, \tilde{\mathbf{q}}_1 + \dots,$$

and an analogous expansion for the pressure $p = p_i + \tilde{p}$. This expansion can lead to divergent results at certain singular points as will be shown later. The local divergence of the boundary layer will not change, however, the general analysis.

According to the boundary-layer assumption, the variation of boundary-layer solution $\tilde{\mathbf{q}}$ in the direction normal to the boundary is of the order $E^{-\frac{1}{2}}$ larger than in the directions parallel to the boundary. Hence it is convenient to introduce the co-ordinate ζ,

$$\zeta = (\mathbf{r}_s - \mathbf{r}).\mathbf{n}E^{-\frac{1}{2}}, \tag{2.5}$$

where $\mathbf{r}_s$ is the position vector on the surface Σ, and $\mathbf{n}$ is the unit vector normal to Σ pointing outward. Using ζ the continuity equation for $\tilde{\mathbf{q}}$ can be written

$$\nabla.\tilde{\mathbf{q}} = -E^{-\frac{1}{2}}\frac{\partial}{\partial\zeta}\tilde{\mathbf{q}}.\mathbf{n} + \mathbf{n}.\nabla\times(\mathbf{n}\times\tilde{\mathbf{q}}) - \mathbf{n}\times\tilde{\mathbf{q}}.\nabla\times\mathbf{n} + \mathbf{n}.\tilde{\mathbf{q}}\nabla.\mathbf{n} = 0. \tag{2.6}$$

The physical aspect of the problem becomes evident in the torque balance

$$2\int(\mathbf{\Omega}\times\mathbf{q})\times\mathbf{r}\,dV + \oint\mathbf{n}\times\mathbf{r}p\,d\Sigma = E\oint(\mathbf{n}.\nabla\mathbf{q}\times\mathbf{r} - 2\mathbf{n}\times\mathbf{q})\,d\Sigma, \tag{2.7}$$

which shows that the pressure as well as the viscous stresses exerted by the boundary can balance the precessional force. In the case of a spherical cavity the pressure term vanishes. Another torque balance can be obtained by extending the integrals over the interior body of the fluid only and neglecting the dissipative term,

$$2\int(\mathbf{\Omega}\times\mathbf{q}_i)\times r\,dV + \oint\mathbf{n}\times\mathbf{r}p_i\,d\Sigma = -\oint\mathbf{q}_i.\mathbf{n}\mathbf{q}_i\times\mathbf{r}\,d\Sigma. \tag{2.8}$$

Since the normal derivative at the boundary is of the order $\epsilon E^{-\frac{1}{2}}$, where ϵ is the amplitude of the boundary-layer flow $\tilde{\mathbf{q}}$, the right-hand sides in (2.7) and (2.8) are of the order $\epsilon\sqrt{E}$. Hence in general part of the Coriolis force is balanced by terms of the order $\sqrt{E}$. Without specifying this part we write the equations for the lowest order of the interior flow in the form

$$\left.\begin{aligned} &\nabla.\mathbf{q}_0 = 0,\\ &2\mathbf{\Omega}\times\mathbf{q}_0 - \sqrt{E}\,\mathbf{F}(\mathbf{q}_0) + \mathbf{q}_0.\nabla\mathbf{q}_0 = -\nabla p_0. \end{aligned}\right\} \tag{2.9}$$

The term $\mathbf{F}(\mathbf{q}_0)$ will appear as an inhomogeneous term in the equation of $\mathbf{q}_1$. The boundary condition for $\mathbf{q}_0$ is $\mathbf{q}_0.\mathbf{n} = 0$ on Σ, since it can be concluded from continuity equation (2.6) that $\mathbf{n}.\tilde{\mathbf{q}}_0$ vanishes.

With the general, as yet not completely determined, solution of (2.9) we enter the equation for the boundary-layer flow $\tilde{\mathbf{q}}_0$

$$2\mathbf{\Omega}\times\tilde{\mathbf{q}}_0 + (\tilde{\mathbf{q}}_0 + \mathbf{q}_0).\nabla\tilde{\mathbf{q}}_0 + \tilde{\mathbf{q}}_0.\nabla\mathbf{q}_0 = \mathbf{n}\frac{\partial}{\partial\zeta}\tilde{p}_1 + \frac{\partial^2}{\partial\zeta^2}\tilde{\mathbf{q}}_0, \tag{2.10}$$

which has to be solved subject to the boundary condition $\tilde{\mathbf{q}}_0 + \mathbf{q}_0 = \mathbf{k}\times\mathbf{r}$. The integration of the continuity equation (2.6) with respect to ζ in general leads to a non-vanishing value of $\tilde{\mathbf{q}}_1.\mathbf{n}$ at the boundary,

$$\tilde{\mathbf{q}}_1.\mathbf{n}|_{\zeta=0} = -\int_0^\infty \mathbf{n}.\nabla\times(\mathbf{n}\times\tilde{\mathbf{q}}_0)\,d\zeta. \tag{2.11}$$

Hence the solution $\mathbf{q}_1$ of the interior equations in the order $\sqrt{E}$,

$$\left.\begin{aligned} &\nabla.\mathbf{q}_1 = 0,\\ &2\mathbf{\Omega}\times\mathbf{q}_1 + \mathbf{q}_0.\nabla\mathbf{q}_1 + \mathbf{q}_1.\nabla\mathbf{q}_0 + \nabla p_1 = -\mathbf{F}(\mathbf{q}_0), \end{aligned}\right\} \tag{2.12}$$

has to satisfy the boundary condition $\mathbf{n}.(\mathbf{q}_1+\tilde{\mathbf{q}}_1) = 0$ on Σ. Since $\sqrt{E}$ usually is sufficiently small, the equations (2.12) do not have to be solved explicitly. The solvability condition for the inhomogeneous linear boundary-value problem (2.12), however, is essential for the determination of the solution $\mathbf{q}_0$. By multiplying the second equation (2.12) with the complete set of solutions of the corresponding adjoint homogeneous problem and integrating it over the contained fluid, a necessary and sufficient condition is obtained for the existence of the solution $\mathbf{q}_1$. This condition provides the additional information necessary for the determination of $\mathbf{q}_0$. It will be shown later that the solvability condition includes the torque balance (2.8) as a special case.

Although the problem in the present formulation can be solved in principle, the non-linear boundary-layer equation prohibits a simple analytical solution. Therefore, we will restrict ourselves to small amplitudes ϵ of the boundary-layer flow and assume, in place of the single expansion (2.4), the double expansion

$$\mathbf{q}_i = \mathbf{q}_0^{(0)}+\epsilon\mathbf{q}_0^{(1)}+\epsilon^2\mathbf{q}_0^{(2)}+\ldots+\sqrt{E}\,\epsilon\mathbf{q}_1^{(1)}+\ldots,\quad \tilde{\mathbf{q}} = \epsilon\tilde{\mathbf{q}}_0^{(1)}+\epsilon^2\tilde{\mathbf{q}}_0^{(2)}+\ldots+\sqrt{E}\,\epsilon\tilde{\mathbf{q}}_1^{(1)}+\ldots, \tag{2.13}$$

with

$$\mathbf{q}_0^{(0)} = \mathbf{k}\times\mathbf{r}. \tag{2.14}$$

A similar expansion has been used by Greenspan & Weinbaum (1965) for the time-dependent spin-up problem of a contained fluid.

In the following section, we will discuss the linear problem explicitly in the case of a spheroidal cavity given by

$$\mathbf{r}^2+\eta\frac{2-\eta}{(1-\eta)^2}(\mathbf{k}.\mathbf{r})^2 = 1. \tag{2.15}$$

We are interested only in the first-order effects due to the eccentricity and assume that the ellipticity η is small compared with one. To avoid another formal expansion in powers of η and because the parameters η and $\sqrt{E}$ occur parallel in the problem according to the torque balance (2.7), we shall treat the dependence of the problem on η on the same level as that on $\sqrt{E}$. In §4 the effects due to the non-linear term $\tilde{\mathbf{q}}_0.\nabla\tilde{\mathbf{q}}_0$ in the boundary-layer equation are considered.

3. The linear boundary-layer problem

Anticipating the result that the solvability condition in the order $\sqrt{E}\,\epsilon$ can be satisfied by a solution $\mathbf{q}_0^{(1)}$ with constant vorticity, we will not start with the most general solution of the equations (2.9), but assume

$$\mathbf{k}\times\mathbf{r}+\epsilon\mathbf{q}_0^{(1)} = \boldsymbol{\omega}\times\mathbf{r} \tag{3.1}$$

with an arbitrary vector $\boldsymbol{\omega}$. This solution does not satisfy the boundary condition $\mathbf{n}.\mathbf{q}_0^{(1)} = 0$ exactly. However, $(\boldsymbol{\omega}-\mathbf{k})\times\mathbf{r}.\mathbf{n}$ is of the order $\epsilon\eta$ and thus can be added in the boundary condition for $\mathbf{q}_1^{(1)}$. The part $\sqrt{E}\,\mathbf{F}$ of the precessional force which cannot be balanced by the pressure and which will be taken into account in the equation of the order $\epsilon\sqrt{E}$ is given by $(\boldsymbol{\Omega}\times\boldsymbol{\omega})\times\mathbf{r}$.

It is convenient to include the term with $\epsilon\mathbf{q}_0^{(1)}$ in the first order of the boundary-layer equations and to write

$$2\boldsymbol{\Omega}\times\tilde{\mathbf{q}}_0^{(1)}+(\boldsymbol{\omega}\times\mathbf{r}).\nabla\tilde{\mathbf{q}}_0^{(1)} = \mathbf{n}\frac{\partial}{\partial\zeta}\tilde{p}_1^{(1)}+\frac{\partial^2}{\partial\zeta^2}\tilde{\mathbf{q}}_0^{(1)}. \tag{3.2}$$

By multiplying (3.2) with $\mathbf{n}\times$ and $-i\mathbf{n}\times(\mathbf{n}\times$ and adding the two results, we obtain

$$\left[\frac{\partial^2}{\partial\zeta^2}-2i(\mathbf{\Omega}+\boldsymbol{\omega}).\mathbf{n}\right](\mathbf{n}\times\tilde{\mathbf{q}}_0^{(1)}+i\tilde{\mathbf{q}}_0^{(1)}) = [\mathbf{n}\times -i\mathbf{n}\times(\mathbf{n}\times\,]((\boldsymbol{\omega}\times\mathbf{r}).\nabla\tilde{\mathbf{q}}_0^{(1)}-\boldsymbol{\omega}\times\tilde{\mathbf{q}}_0^{(1)}). \tag{3.3}$$

The solution of this equation which satisfies the boundary condition

$$\epsilon\tilde{\mathbf{q}}_0^{(1)} = (\mathbf{k}-\boldsymbol{\omega})\times\mathbf{r}$$

is given by

$$\mathbf{n}\times\tilde{\mathbf{q}}_0^{(1)}+i\tilde{\mathbf{q}}_0^{(1)} = (\mathbf{n}\times(\boldsymbol{\omega}_+\times\mathbf{r})+i\boldsymbol{\omega}_+\times\mathbf{r})\exp(-\kappa_+\zeta)+(\mathbf{n}\times(\boldsymbol{\omega}_-\times\mathbf{r})+i\boldsymbol{\omega}_-\times\mathbf{r})$$
$$\times\exp(-\kappa_-\zeta)-\frac{1}{\epsilon}\left(1-\frac{\mathbf{k}.\boldsymbol{\omega}}{\omega^2}\right)(\mathbf{n}\times(\boldsymbol{\omega}\times\mathbf{r})+i\boldsymbol{\omega}\times\mathbf{r})\exp(-\kappa\zeta), \tag{3.4}$$

with
$$\omega_\pm = -\frac{\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{k})}{2\omega^2\epsilon}\pm i\frac{\boldsymbol{\omega}\times\mathbf{k}}{2\omega\epsilon}.$$

As the appropriate definition for the amplitude ϵ of the boundary velocity, we have assumed $\epsilon^2 = (\mathbf{k}-\boldsymbol{\omega})^2$. The coefficients κ_+, κ_-, κ are determined by the relations

$$\left.\begin{aligned}\kappa_\pm^2-2i(\mathbf{\Omega}+\boldsymbol{\omega}).\mathbf{n} &= \pm i\omega,\\ \kappa^2-2i(\mathbf{\Omega}+\boldsymbol{\omega}).\mathbf{n} &= 0,\end{aligned}\right\} \tag{3.5}$$

which are derived from (3.3) using the formula

$$[(\boldsymbol{\omega}\times\mathbf{r}).\nabla-\boldsymbol{\omega}\times\,](\mathbf{n}\times(\boldsymbol{\omega}_\pm\times\mathbf{r})+i\boldsymbol{\omega}_\pm\times\mathbf{r}) = \pm i\omega(\mathbf{n}\times(\boldsymbol{\omega}_\pm\times\mathbf{r})+i\boldsymbol{\omega}_\pm\times\mathbf{r}) \tag{3.6}$$

and neglecting terms of the order $|(\mathbf{\Omega}\times\boldsymbol{\omega})\times\mathbf{r}|$ in consistency with the approximation. According to the boundary-layer assumption, the root with positive real part has to be chosen for κ_+, κ_-, and κ.

The influx from the boundary is given according to relation (2.11) by

$$\mathbf{n}.\tilde{\mathbf{q}}_1^{(1)}|_{\zeta=0} = \mathbf{n}.\nabla\times\left\{\sum\frac{\mathbf{n}\times(\boldsymbol{\omega}_\pm\times\mathbf{r})+i\boldsymbol{\omega}_\pm\times\mathbf{r}}{2\kappa_\pm}-\left(1-\frac{\mathbf{k}.\boldsymbol{\omega}}{\omega^2}\right)\frac{\mathbf{n}\times(\boldsymbol{\omega}\times\mathbf{r})+i\boldsymbol{\omega}\times\mathbf{r}}{\epsilon 2\kappa}\right\}+\text{c.c.}, \tag{3.7}$$

where the summation is to be extended over the two possibilities $+$ and $-$ of the index and c.c. stands for the complex conjugate. The expression (3.7) enters the boundary condition for the equations of the interior in the order $\sqrt{E}\,\epsilon$ together with the unsatisfied part of the boundary condition in the order E^0,

$$2\mathbf{\Omega}\times\mathbf{q}_1^{(1)}+(\boldsymbol{\omega}\times\mathbf{r}).\nabla\mathbf{q}_1^{(1)}+\boldsymbol{\omega}\times\mathbf{q}_1^{(1)} = -\nabla p_1^{(1)}-(\mathbf{\Omega}\times\boldsymbol{\omega})\times\mathbf{r}(1/\sqrt{E}\,\epsilon), \tag{3.8}$$
$$\mathbf{n}.\mathbf{q}_1^{(1)} = -\mathbf{n}.\tilde{\mathbf{q}}_1^{(1)}|_{\zeta=0}-(1/\sqrt{E}\,\epsilon)\,\boldsymbol{\omega}\times\mathbf{r}.\mathbf{n}\quad\text{on }\Sigma.$$

By multiplying (3.8) with the solutions $\mathbf{q}_H$ of the corresponding homogeneous problem

$$\left.\begin{aligned}2\mathbf{\Omega}\times\mathbf{q}_H+(\boldsymbol{\omega}\times\mathbf{r}).\nabla\mathbf{q}_H+\boldsymbol{\omega}\times\mathbf{q}_H &= -\nabla p_H,\\ \mathbf{n}.\mathbf{q}_H = 0\quad\text{on }\Sigma,&\end{aligned}\right\} \tag{3.9}$$

we obtain the solvability condition for the problem (3.8)

$$\oint p_H\mathbf{n}.\mathbf{q}_1^{(1)}d\Sigma = -\frac{1}{\sqrt{E}\,\epsilon}\int\mathbf{q}_H.[(\mathbf{\Omega}\times\boldsymbol{\omega})\times\mathbf{r}]dV. \tag{3.10}$$

A solution of (3.8) exists when relation (3.10) is satisfied for all solutions $\mathbf{q}_H$ of the problem (3.9). An approximate solution of (3.9) is given by

$$\mathbf{q}_H \approx \boldsymbol{\omega}^* \times \mathbf{r}, \quad p_H \approx (\boldsymbol{\omega}^* \times \mathbf{r}).(\boldsymbol{\omega} \times \mathbf{r}) \tag{3.11}$$

with an arbitrary vector $\boldsymbol{\omega}^*$. Since this solution does not fulfil equation and boundary condition (3.9) exactly, we find by repeating the derivation of the solvability condition with the choice (3.11) for $\mathbf{q}_H$ that the terms

$$2\int \boldsymbol{\Omega} \times (\boldsymbol{\omega}^* \times \mathbf{r}).\mathbf{q}_{\mathrm{I}}^{(1)} dV - \boldsymbol{\omega}^*.\oint \mathbf{r} \times \mathbf{n} p_{\mathrm{I}}^{(1)} d\Sigma$$

have to be added on the right side of (3.10), in which $\mathbf{q}_H$ and p_H have been replaced according to (3.11). With this addition we rewrite the condition (3.10) in the following form using the boundary condition (3.8) for $\mathbf{q}_{\mathrm{I}}^{(1)}$:

$$\epsilon\sqrt{E}\,\boldsymbol{\omega}^*.\left\{\oint \tilde{\mathbf{q}}_{\mathrm{I}}^{(1)}.\mathbf{n}|_{\zeta=0}(\boldsymbol{\omega}\times\mathbf{r})\times\mathbf{r}\,d\Sigma - 2\int(\boldsymbol{\Omega}\times\mathbf{q}_{\mathrm{I}}^{(1)})\times\mathbf{r}\,dV\right\}$$
$$= \boldsymbol{\omega}^*.\left\{\oint \mathbf{n}\times\mathbf{r}\left(\frac{|\boldsymbol{\omega}\times\mathbf{r}|^2}{2} + \epsilon\sqrt{E}\,p_{\mathrm{I}}^{(1)}\right)d\Sigma + \int[(\boldsymbol{\Omega}\times\boldsymbol{\omega})\times\mathbf{r}]\times\mathbf{r}\,dV\right\}. \tag{3.12}$$

Since $\boldsymbol{\omega}^*$ is an arbitrary vector, the comparison with (2.8) shows that (3.12) describes the torque balance for $\mathbf{q}_i = \boldsymbol{\omega}\times\mathbf{r} + \epsilon\sqrt{E}\,\mathbf{q}_{\mathrm{I}}^{(1)}$.

The vector multiplied by $\boldsymbol{\omega}^*$ on the right side of (3.12) is of the order $\epsilon\sqrt{E}$ according to the torque balance. When $\boldsymbol{\omega}^*$ is chosen equal to $\boldsymbol{\omega}$, the right side of (3.12) vanishes in the special case $\eta = 0$ of a spherical cavity. Hence in the general case the right side is of the order $\epsilon\sqrt{E}\,\epsilon\eta$ for $\boldsymbol{\omega}^* = \boldsymbol{\omega}$ since the ellipticity enters the problem in the combination $\epsilon\eta$. The left side of (3.12) can be evaluated for $\boldsymbol{\omega}^* = \boldsymbol{\omega}$ in the following way:

$$\boldsymbol{\omega}.\oint \tilde{\mathbf{q}}_{\mathrm{I}}^{(1)}.\mathbf{n}|_{\zeta=0}(\boldsymbol{\omega}\times\mathbf{r})\times\mathbf{r}\,d\Sigma - 2\boldsymbol{\omega}.\int(\boldsymbol{\Omega}\times\mathbf{q}_{\mathrm{I}}^{(1)})\times\mathbf{r}\,dV$$
$$= \oint \mathbf{n}.\nabla\times\int_0^\infty \mathbf{n}\times\tilde{\mathbf{q}}_0^{(1)}d\zeta\,|\boldsymbol{\omega}\times\mathbf{r}|^2 d\Sigma + \oint \mathbf{q}_{\mathrm{I}}^{(1)}.\mathbf{n}\frac{\boldsymbol{\Omega}.\boldsymbol{\omega}}{\omega^2}|\boldsymbol{\omega}\times\mathbf{r}|^2 d\Sigma$$
$$- \int(\boldsymbol{\Omega}\times\boldsymbol{\omega})\times\mathbf{r}.\mathbf{q}_{\mathrm{I}}^{(1)}dV$$
$$= -\frac{2}{\epsilon}\left(1+\frac{\boldsymbol{\Omega}.\boldsymbol{\omega}}{\omega^2}\right)\left(1-\frac{\mathbf{k}.\boldsymbol{\omega}}{\omega^2}\right)\oint\left(\frac{\mathbf{n}\times(\boldsymbol{\omega}\times\mathbf{r})+i\boldsymbol{\omega}\times\mathbf{r}}{2\kappa}+\text{c.c.}\right).\mathbf{n}\times(\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}))\,d\Sigma$$
$$+\text{terms of the order } \epsilon.\max(\sqrt{E},|\eta|). \tag{3.13}$$

Since the last integral in (3.13) is of the order 1, the solvability condition (3.12) with $\boldsymbol{\omega}^* = \boldsymbol{\omega}$ requires

$$\omega^2 = \mathbf{k}.\boldsymbol{\omega}, \tag{3.14}$$

and hence

$$\epsilon^2 = 1-\omega^2, \tag{3.15}$$

where terms not larger than of the order $\epsilon^2.\max(\sqrt{E},|\eta|)$ have been neglected. The physical interpretation of (3.14) is that in a stationary state the component of the angular velocity of the container parallel to the angular velocity $\boldsymbol{\omega}$ of the fluid has to be equal to ω in order that no spin-up process occur. The relation

(3.14) determines $\boldsymbol{\omega}$ completely in the case where Ω is of the order 1. Since $|\boldsymbol{\omega}\times\boldsymbol{\Omega}|$ is of the order $\epsilon.\max(\sqrt{E}, |\eta|)$, $\boldsymbol{\omega}$ has to coincide with the precession axis to the approximation in which the equations have been considered. $\boldsymbol{\omega}$ will be parallel or antiparallel to $\boldsymbol{\Omega}$, depending on the sign of $\mathbf{k}.\boldsymbol{\Omega}$ according to (3.14).

Additional information is needed to determine $\boldsymbol{\omega}$ when Ω is a small quantity like $\sqrt{E}$ or η. In this case the term with $\boldsymbol{\Omega}\times\mathbf{q}_1^{(1)}$ can be neglected in (3.12) as well as the term with $p_1^{(1)}$, and two independent relations are given by the real and imaginary parts of (3.12) when $\boldsymbol{\omega}^*$ is chosen equal to $\boldsymbol{\omega}_+$. The right side of (3.12) yields

$$\int(\boldsymbol{\omega}_+\times\mathbf{r}).\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})\,dV-\int(\boldsymbol{\omega}_+\times\mathbf{r}).(\boldsymbol{\Omega}\times\boldsymbol{\omega})\times\mathbf{r}\,dV$$
$$=\frac{4\pi}{15}2i\omega\boldsymbol{\Omega}.\boldsymbol{\omega}_++\frac{4\pi}{15}i\eta\epsilon\omega\boldsymbol{\omega}.\mathbf{k}. \qquad (3.16)$$

Consistent with the approximation we have neglected higher-order terms by extending the first integral over a spherical volume. In the same approximation the integral on the left side of (3.12) yields

$$\oint(\boldsymbol{\omega}\times\mathbf{r}).(\boldsymbol{\omega}_+\times\mathbf{r})\mathbf{n}.\tilde{\mathbf{q}}_1^{(1)}d\Sigma$$
$$=\oint\left(\frac{\mathbf{n}\times(\boldsymbol{\omega}_-\times\mathbf{r})+i\boldsymbol{\omega}_-\times\mathbf{r}}{2\kappa_-}+\frac{\mathbf{n}\times(\boldsymbol{\omega}_-\times\mathbf{r})-i\boldsymbol{\omega}_-\times\mathbf{r}}{2\kappa_+}\right)$$
$$.\{\mathbf{n}\times(2\boldsymbol{\omega}\times(\boldsymbol{\omega}_+\times\mathbf{r})+i\omega\boldsymbol{\omega}_+\times\mathbf{r})\}d\Sigma$$
$$=\frac{\pi\sqrt{\omega}}{2\sqrt{2}}\left((1-i)\frac{6\sqrt{27}}{35}+(1+i)\frac{38}{35}\right). \qquad (3.17)$$

Using the results (3.16) and (3.17) the solvability condition (3.12) assumes the form

$$\boldsymbol{\Omega}.\left(\frac{\boldsymbol{\omega}\times\mathbf{k}}{\epsilon^2}+i\frac{\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{k})}{\omega\epsilon^2}\right)=2{\cdot}62(E\omega)^{\frac{1}{2}}+i(0{\cdot}259(E\omega)^{\frac{1}{2}}+\eta\omega(\boldsymbol{\omega}.\mathbf{k})). \quad (3.18)$$

This relation together with equation (3.14) determines

$$\boldsymbol{\omega}=\mathbf{k}\omega^2+\frac{\omega^2(\mathbf{k}\times\boldsymbol{\Omega}2{\cdot}62(E\omega)^{\frac{1}{2}})+\omega^2\mathbf{k}\times(\boldsymbol{\Omega}\times\mathbf{k})(0{\cdot}259(E/\omega)^{\frac{1}{2}}+\eta\omega^2+\boldsymbol{\Omega}).\mathbf{k})}{(0{\cdot}259(E/\omega)^{\frac{1}{2}}+\eta\omega^2+\boldsymbol{\Omega}.\mathbf{k})^2+2{\cdot}62^2E\omega}. \qquad (3.19)$$

This result holds to the order ϵ^2 since the only term, $\tilde{\mathbf{q}}.\nabla\tilde{\mathbf{q}}$, of the order ϵ^2 which has not yet been considered does not affect the direction of $\boldsymbol{\omega}$ because it is either axisymmetric or doubly periodic around the axis $\boldsymbol{\omega}$. Thus the critical circles $\kappa_+=0$ and $\kappa_-=0$, at which the boundary-layer thickness tends to infinity, occur with respect to the $\boldsymbol{\omega}$-axis instead of the $\mathbf{k}$-axis as in the linear theory. The mathematical problem caused by this singularity has been discussed by Stewartson & Roberts (1963). They conclude that the boundary-layer thickness changes from the order $E^{\frac{1}{2}}$ to the order $E^{\frac{2}{5}}$ at the critical circles and that accordingly the influx into the interior from this region is small compared with the influx from the rest of the boundary.

Since the second term on the right side of (3.19) is of the order ϵ, it is consistent to replace ω by 1 in this term, in which case the expression becomes identical with the results of Roberts & Stewartson (1965). Qualitatively, however, the direction of $\boldsymbol{\omega}$ is described by (3.19) to the order ϵ^3 since the basic balance (3.18) is not changed by terms of this order which have been neglected. For this reason $\boldsymbol{\omega}$ does not diverge in the limit when $\sqrt{E}$ and $\boldsymbol{\Omega}.\mathbf{k}+\eta$ tend to zero. The condition that ϵ is a small quantity can be derived from (3.19) in terms of the given parameter of the problem,

$$\min\left(\frac{\Omega\sin\alpha}{\sqrt{E}}, \frac{\alpha}{|\boldsymbol{\Omega}.\mathbf{k}+\eta|}, \frac{\Omega}{|\eta|}\right) \ll 1, \tag{3.20}$$

where α is the angle between $\boldsymbol{\Omega}$ and $\mathbf{k}$. Since the non-linear effects up to the order ϵ^2 have been taken into account, this inequality does not have to be satisfied in a very strong sense.

The direction of $\boldsymbol{\omega}$ can be visualized best in the case of the spherical cavity. Equation (3.18) shows that for $\eta = 0$ the direction of $\boldsymbol{\omega}$ varies as a function of $\sqrt{E}/\Omega$ along a cone of nearly half-circular shape as far as 0·259 is small compared with 2·62. The edges of the cone corresponding to the end-points of the half-circle are given by $\boldsymbol{\Omega}$ and $\mathbf{k}$.

4. The non-linear part of the boundary-layer problem

In the last section we did not have to use the solvability condition in its most general form because we began the analysis with the special solution (3.1). The general solution in the form of a differential rotation would not have led to a different result. With the choice of the same solution for $\mathbf{q}_H$, the solvability condition (3.10) cannot be satisfied unless the solution reduces to the special form (3.1). In this section, however, it will be shown that due to the action of the non-linear term in the boundary-layer equation, a differential rotation is induced in the interior. Since the eccentricity has only secondary importance in the non-linear part of the problem, we are not losing generality by restricting our attention to the case of the spherical cavity.

The inviscid equations for the interior

$$\left.\begin{aligned} &\nabla.\mathbf{q}_0^{(2)} = 0,\\ &2\boldsymbol{\Omega}\times\mathbf{q}_0^{(2)} + (\boldsymbol{\omega}\times\mathbf{r}).\nabla\mathbf{q}_0^{(2)} + \boldsymbol{\omega}\times\mathbf{q}_0^{(2)} + \nabla p_0^{(2)} = 0,\end{aligned}\right\} \tag{4.1}$$

allow a solution of the form

$$\mathbf{q}_0^{(2)} = \boldsymbol{\omega}\times\mathbf{r}f\left(\frac{|\boldsymbol{\omega}\times\mathbf{r}|}{\omega}\right) \tag{4.2}$$

with arbitrary function f, since the component of $\boldsymbol{\Omega}$ normal to $\boldsymbol{\omega}$ is of the order $\epsilon\sqrt{E}$ and can be neglected in this order. The boundary-layer problem in the second order is given by

$$\begin{aligned} 2\left(\frac{\boldsymbol{\Omega}.\boldsymbol{\omega}}{\omega^2}+1\right)\boldsymbol{\omega}\times\tilde{\mathbf{q}}_0^{(2)} + (\boldsymbol{\omega}\times\mathbf{r}).\nabla\tilde{\mathbf{q}}_0^{(2)} - \boldsymbol{\omega}\times\tilde{\mathbf{q}}_0^{(2)} - \mathbf{n}\frac{\partial}{\partial\zeta}\tilde{p}_1^{(2)} - \frac{\partial^2}{\partial\zeta^2}\tilde{\mathbf{q}}_0^{(2)} \\ = -\tilde{\mathbf{q}}_0^{(1)}.\nabla\tilde{\mathbf{q}}_0^{(1)} + (\tilde{\mathbf{q}}_1^{(1)}+\mathbf{q}_1^{(1)}).\mathbf{n}\frac{\partial}{\partial\zeta}\tilde{\mathbf{q}}_0^{(1)}, \\ \tilde{\mathbf{q}}_0^{(2)} + \mathbf{q}_0^{(2)} = 0 \quad \text{on } \Sigma. \end{aligned} \tag{4.3}$$

The arbitrary function f will be determined by the solvability condition for the interior equations in the order $\epsilon^2\sqrt{E}$. These equations have the following form, if we restrict ourselves for the moment to that part of the problem which is axisymmetric with respect to $\boldsymbol{\omega}$,

$$\nabla . \mathbf{q}_1^{(2)} = 0, \quad 2\left(\frac{\boldsymbol{\Omega}.\boldsymbol{\omega}}{\omega^2}+1\right)\boldsymbol{\omega}\times\mathbf{q}_1^{(2)}+\nabla p_1^{(2)} = 0, \tag{4.4}$$

with the boundary condition $\mathbf{n}.\mathbf{q}_1^{(2)}+\mathbf{n}.\tilde{\mathbf{q}}_1^{(2)}|_{\zeta=0} = 0$. A consequence of (4.4) is

$$\boldsymbol{\omega}.\nabla\mathbf{q}_1^{(2)} = 0. \tag{4.5}$$

Thus the part of the influx into the interior, $\mathbf{n}.\tilde{\mathbf{q}}_1^{(2)}|_{\zeta=0}$, which is symmetric with respect to the axis and to the equatorial plane, has to vanish at any distance from the axis. This condition alone will be sufficient to determine f, and we therefore shall solve only the axisymmetric part of the problem (4.3).

Since the non-axisymmetric part of the inhomogeneity in (4.3) is doubly periodic around the axis, the corresponding part of the solvability condition for the interior equations in the order $\epsilon^2\sqrt{E}$ is generally fulfilled. For Ω of the order $\sqrt{E}$ or for negative $\boldsymbol{\Omega}.\boldsymbol{\omega}$, this follows directly from the fact that no doubly periodic solutions of the homogeneous problem (3.9) can exist (see, for example, Greenspan 1964).

We divide the solution of the boundary-layer problem (4.3) into two steps. First, we obtain the solution $\tilde{\mathbf{q}}_A$ of the homogeneous equation together with the inhomogeneous boundary condition. Then we add the solution $\tilde{\mathbf{q}}_B$ of the inhomogeneous equation obeying the boundary condition $\tilde{\mathbf{q}}_0^{(2)} = 0$ on Σ. Since the axisymmetric part of the inhomogeneity in (4.3) is symmetric with respect to the equatorial plane, the influx into the interior has to vanish,

$$\mathbf{n}.\tilde{\mathbf{q}}_1^{(2)}|_{\zeta=0} = -\mathbf{n}.\nabla\times\int_0^\infty \mathbf{n}\times\tilde{\mathbf{q}}_0^{(2)}d\zeta = -\mathbf{n}.\nabla\times\int_0^\infty \mathbf{n}\times(\tilde{\mathbf{q}}_A+\tilde{\mathbf{q}}_B)d\zeta = 0. \tag{4.6}$$

This condition will be sufficient to determine the function f.

The solution $\tilde{\mathbf{q}}_A$ is easily obtained by the same method as used in §3,

$$\mathbf{n}\times\tilde{\mathbf{q}}_A+i\tilde{\mathbf{q}}_A = -[\mathbf{n}\times(\boldsymbol{\omega}\times\mathbf{r})+i\boldsymbol{\omega}\times\mathbf{r}]f\left(\frac{|\boldsymbol{\omega}\times\mathbf{r}|}{\omega}\right)\exp(-\kappa\zeta). \tag{4.7}$$

The corresponding influx into the interior is given by

$$-\mathbf{n}.\nabla\times\int_0^\infty \mathbf{n}\times\tilde{\mathbf{q}}_A d\zeta = \tfrac{1}{2}\mathbf{n}.\nabla\times\left\{\left(\mathbf{n}\times(\boldsymbol{\omega}\times\mathbf{r})+\frac{\mathbf{n}.\boldsymbol{\omega}}{|\mathbf{n}.\boldsymbol{\omega}|}\boldsymbol{\omega}\times\mathbf{r}\right)\right.$$
$$\left. .f\left(\frac{|\boldsymbol{\omega}\times\mathbf{r}|}{\omega}\right)\left(|\mathbf{n}.\boldsymbol{\omega}|\left\{1+\frac{\boldsymbol{\omega}.\boldsymbol{\Omega}}{\omega^2}\right\}\right)^{-\frac{1}{2}}\right\}. \tag{4.8}$$

The solution of the equation for $\tilde{\mathbf{q}}_B$

$$\left[\frac{\partial^2}{\partial\zeta^2}-2i\left(\frac{\boldsymbol{\Omega}.\boldsymbol{\omega}}{\omega^2}+1\right)\boldsymbol{\omega}.\mathbf{n}\right](\mathbf{n}\times\tilde{\mathbf{q}}_B+i\tilde{\mathbf{q}}_B) = [-\mathbf{n}\times+i\mathbf{n}\times(\mathbf{n}\times](\mathbf{J}_I+\mathbf{J}_{II}) \tag{4.9}$$

is more laborious, and we shall assume for simplicity that $\mathbf{\Omega}.\boldsymbol{\omega}$ can be neglected in comparison with ω^2. In this case the inhomogeneous terms are given by

$$[-\mathbf{n}\times+i\mathbf{n}\times(\mathbf{n}\times]\mathbf{J}_I \equiv [\mathbf{n}\times-i\mathbf{n}\times(\mathbf{n}\times]\tilde{\mathbf{q}}_0^{(1)}.\nabla\tilde{\mathbf{q}}_0^{(1)}$$

$$= -\{\Sigma[\boldsymbol{\omega}_\pm\times\mathbf{r}-i\mathbf{n}\times(\boldsymbol{\omega}_\pm\times\mathbf{r})]\boldsymbol{\omega}_\mp.\mathbf{n}\}\exp(-(\kappa_++\kappa_-)\zeta)$$

$$-\Sigma[\boldsymbol{\omega}_\pm\times\mathbf{r}-i\mathbf{n}\times(\boldsymbol{\omega}_\pm\times\mathbf{r})]\boldsymbol{\omega}_\mp.\mathbf{n}\frac{i\zeta}{2\kappa_\pm}(\exp[-(\kappa_\pm+\kappa_\mp)\zeta](\mp\omega-\boldsymbol{\omega}.\mathbf{n})$$

$$+\exp[-(\kappa_\pm+\bar{\kappa}_\pm)\zeta](\mp\omega+\boldsymbol{\omega}.\mathbf{n})),$$

$$[-\mathbf{n}\times+i\mathbf{n}\times(\mathbf{n}\times]\mathbf{J}_{II} \equiv [-\mathbf{n}\times+i\mathbf{n}\times(\mathbf{n}\times]\mathbf{n}.(\tilde{\mathbf{q}}_1^{(1)}+\mathbf{q}_1^{(1)})\frac{\partial}{\partial\zeta}\tilde{\mathbf{q}}_0^{(1)}$$

$$\Sigma = i\kappa_\pm[\boldsymbol{\omega}_\pm\times\mathbf{r}-i\mathbf{n}\times(\boldsymbol{\omega}_\pm\times\mathbf{r})]\exp(-\kappa_\pm\zeta)\boldsymbol{\omega}_\mp.\mathbf{n}\left\{\frac{\exp(-\kappa_\mp\zeta)-1}{2\kappa_\mp^3}3(\boldsymbol{\omega}.\mathbf{n}\mp\omega)\right.$$

$$-\frac{\zeta\exp(-\kappa_\mp\zeta)}{2\kappa_\mp^2}(\boldsymbol{\omega}.\mathbf{n}\pm\omega)+\frac{\exp(-\bar{\kappa}_\pm\zeta)-1}{2\bar{\kappa}_\pm^3}3(\boldsymbol{\omega}.\mathbf{n}\pm\omega)$$

$$\left.-\frac{\zeta\exp(-\bar{\kappa}_\pm\zeta)}{2\bar{\kappa}_\pm^2}(\boldsymbol{\omega}.\mathbf{n}\mp\omega)\right\}.$$

Using the relation

$$\boldsymbol{\omega}_\pm.\mathbf{n}(\boldsymbol{\omega}_\mp\times\mathbf{n}) = -\boldsymbol{\omega}\times\mathbf{n}\frac{\mathbf{n}.\boldsymbol{\omega}}{4\omega^2}\pm\frac{i}{4\omega}\mathbf{n}\times(\boldsymbol{\omega}\times\mathbf{n}) \tag{4.10}$$

we obtain the solution of (4.9) satisfying the homogeneous boundary condition $\tilde{\mathbf{q}}_B = 0$,

$$\mathbf{n}\times\tilde{\mathbf{q}}_B+i\tilde{\mathbf{q}}_B = -\frac{1}{4\omega^2}[\mathbf{n}\times(\boldsymbol{\omega}\times\mathbf{r})+i\boldsymbol{\omega}\times\mathbf{r}]$$

$$.\left(\frac{\exp[-(\kappa_++\kappa_-)\zeta]-\exp[-\kappa\zeta]}{(\kappa_++\kappa_-)^2-\kappa^2}\left(2i\mathbf{n}.\boldsymbol{\omega}+\tfrac{3}{2}\Sigma\frac{\kappa_\pm(\mathbf{n}.\boldsymbol{\omega}\mp\omega)^2}{\kappa_\mp^3}\right.\right.$$

$$\left.-\frac{(\kappa_+^2-\kappa_-^2)^2}{(\kappa_++\kappa_-)^2-\kappa^2}\frac{(\mathbf{n}.\boldsymbol{\omega})^2-\omega^2}{\kappa_+^2\kappa_-^2}\right)$$

$$-\frac{\zeta\exp[-(\kappa_++\kappa_-)\zeta]}{(\kappa_++\kappa_-)^2-\kappa^2}\frac{(\kappa_+^2-\kappa_-^2)(\kappa_+-\kappa_-)}{2\kappa_+^2\kappa_-^2}[(\mathbf{n}.\boldsymbol{\omega})^2-\omega^2]$$

$$-\Sigma\tfrac{3}{2}\kappa_\pm\left\{\frac{\exp[-\kappa_\pm\zeta]-\exp[-\kappa\zeta]}{\kappa_\pm^2-\kappa^2}\left(\frac{(\boldsymbol{\omega}.\mathbf{n}\mp\omega)^2}{\kappa_\mp^3}+\frac{(\boldsymbol{\omega}.\mathbf{n})^2-\omega^2}{\bar{\kappa}_\pm^3}\right)\right.$$

$$\left.\left.-\frac{\exp[-(\kappa_\pm+\bar{\kappa}_\pm)\zeta]-\exp[-\kappa\zeta]}{(\kappa_\pm+\bar{\kappa}_\pm)^2-\kappa^2}\frac{(\boldsymbol{\omega}.\mathbf{n})^2-\omega^2}{\bar{\kappa}_\pm^3}\right\}\right).$$

The influx into the interior corresponding to this solution is

$$-\int_0^\infty\mathbf{n}.\nabla\times(\mathbf{n}\times\tilde{\mathbf{q}}_B)d\zeta = \mathbf{n}.\nabla\times(\mathbf{n}\times(\boldsymbol{\omega}\times\mathbf{r}))\frac{D+\bar{D}}{8\omega^2|\mathbf{n}.\boldsymbol{\omega}|^{\frac{1}{2}}}$$

$$-\mathbf{n}.\nabla\times(\boldsymbol{\omega}\times\mathbf{r})\frac{D-\bar{D}}{i8\omega^2|\mathbf{n}.\boldsymbol{\omega}|^{\frac{1}{2}}} \tag{4.11}$$

with

$$D = -\frac{2i\mathbf{n}.\boldsymbol{\omega}+\frac{3}{2}\sum(\mathbf{n}.\boldsymbol{\omega}\mp\omega)^2\kappa_{\pm}\kappa_{\mp}^{-3}-(\kappa_{+}^{2}-\kappa_{-}^{2})^2[(\mathbf{n}.\boldsymbol{\omega})^2-\omega^2][(\kappa_{+}+\kappa_{-})^2-\kappa^2]^{-1}\kappa_{+}^{-2}\kappa_{-}^{-2}}{(\kappa_{+}+\kappa_{-}+\kappa)(\kappa_{+}+\kappa_{-})(1+i\mathbf{n}.\boldsymbol{\omega}/|\mathbf{n}.\boldsymbol{\omega}|)}$$
$$-\frac{[(\mathbf{n}.\boldsymbol{\omega})^2-\omega^2](\kappa_{+}-\kappa_{-})^2|\mathbf{n}.\boldsymbol{\omega}|^{\frac{1}{2}}}{[(\kappa_{+}+\kappa_{-})^2-\kappa^2]\,2\kappa_{+}^{2}\kappa_{-}^{2}(\kappa_{+}+\kappa_{-})}+\sum\tfrac{3}{2}\kappa_{\pm}\frac{(\mathbf{n}.\boldsymbol{\omega}\mp\omega)^2\kappa_{\mp}^{-3}+[(\mathbf{n}.\boldsymbol{\omega})^2-\omega^2]\bar{\kappa}_{\pm}^{-3}}{(\kappa_{\pm}+\kappa)\kappa_{\pm}(1+i\mathbf{n}.\boldsymbol{\omega}/|\mathbf{n}.\boldsymbol{\omega}|)}$$
$$+\frac{3}{2}\frac{\omega^2-(\mathbf{n}.\boldsymbol{\omega})^2}{1+i\mathbf{n}.\boldsymbol{\omega}/|\mathbf{n}.\boldsymbol{\omega}|}\sum\frac{\kappa_{\pm}}{\bar{\kappa}_{\pm}^{3}(\kappa_{\pm}+\bar{\kappa}_{\pm}+\kappa)(\kappa_{\pm}+\bar{\kappa}_{\pm})}.$$

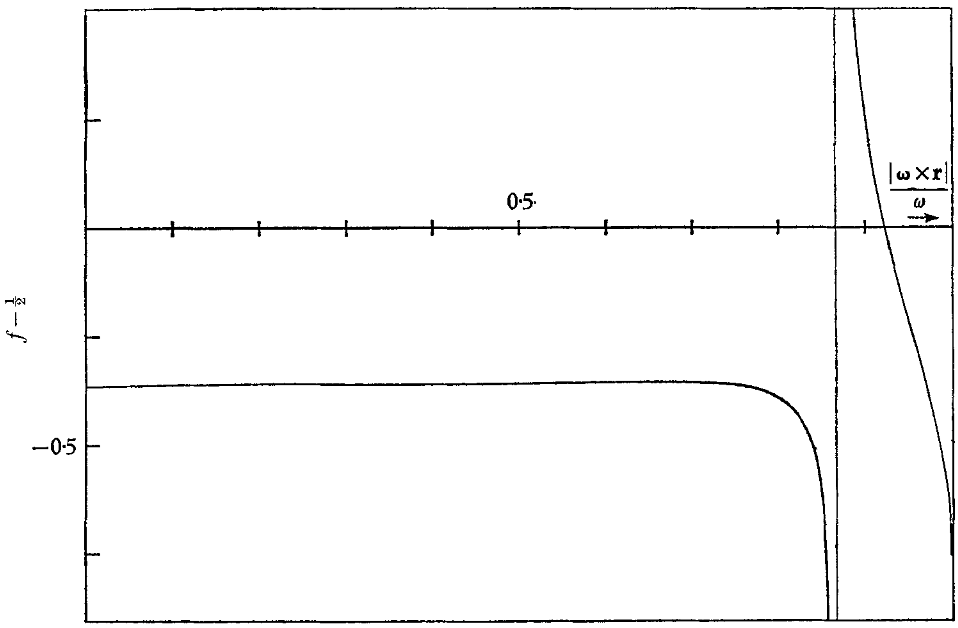

FIGURE 1. The difference of the angular velocity between the fluid and the spherical cavity divided by ϵ^2 as a function of the distance from the axis.

Since the first term on the right side of (4.11) vanishes for any function D, only the imaginary part of D has to be calculated. According to condition (4.6) the function $f(|\boldsymbol{\omega}\times\mathbf{r}|/\omega)$ is given by

$$f\left(\frac{|\boldsymbol{\omega}\times\mathbf{r}|}{\omega}\right)=\frac{D-\bar{D}}{i8\omega^2|\mathbf{n}.\boldsymbol{\omega}|^{\frac{1}{2}}}\frac{|\mathbf{n}.\boldsymbol{\omega}|}{\mathbf{n}.\boldsymbol{\omega}}. \tag{4.12}$$

In figure 1 the function $f-\frac{1}{2}$ is plotted. Since from the linear part of the problem ω is determined to the second order by

$$\omega = 1-\tfrac{1}{2}\epsilon^2, \tag{4.13}$$

the figure shows the difference between the angular velocities of the fluid and the container divided by ϵ^2. Due to the critical circle of the linear boundary solution, at which the boundary-layer thickness becomes divergent, a similar behaviour is shown by the function f. Since this divergence, however, is only of the order

$(\mathbf{n}.\boldsymbol{\omega}-\frac{1}{2}\omega)^{-\frac{1}{2}}$, we expect that the differential rotation will show a smooth profile due to the viscous effects in the interior which have been neglected in this analysis. In the limit of vanishing E the profile f will be approached asymptotically. Experimental evidence reported in the forthcoming paper by Malkus (1968) supports this contention.

5. Steady fluid flow due to a tidal bulge

In this section we consider the problem of the steady flow of a homogeneous fluid inside a spheroidal shell which is deformed by time-independent forces. We assume that the shell is spinning with the constant angular velocity $\boldsymbol{\omega}_s$ relative to an inertial system and that the deformation is time-independent with respect to the same system. The characteristic features of this problem can be exhibited by considering the simple case of a rotating spherical shell which is deformed into a spheroidal shell with negative ellipticity η^* and an axis of symmetry defined by the constant unit vector $\mathbf{t}$. Because of the geophysical significance of the problem, such a deformation is called a tidal bulge. Using the non-dimensional notation introduced in §2, the velocity field with constant vorticity, which satisfies the boundary condition

$$\mathbf{q}.\mathbf{n} = 0 \quad \text{on } \Sigma, \tag{5.1}$$

is given by

$$\mathbf{q}_0^{(0)} = \mathbf{k}\times\mathbf{r} - \frac{\eta^*(1-\eta^*/2)}{(1+\eta^*)(1-\eta^*)^2}\nabla\mathbf{t}.\mathbf{r}(\mathbf{t}\times\mathbf{k}).\mathbf{r}. \tag{5.2}$$

The boundary condition for the tangential component of the velocity depending on the details of the tidal motion of the shell will not be completely satisfied by the flow (5.2) in general. A boundary-layer flow of the order η^* is induced and, due to the influx from the boundary, a change of the same order in the interior flow will be the result. In the following, however, we will neglect this effect†, assuming that η^* is very small compared with unity.

Another boundary-layer problem of different order of magnitude is induced by the fact that the solution (5.2) satisfies the equations of motion only to the approximation in which the term

$$\eta^*\mathbf{t}.\mathbf{k}(\mathbf{t}\times\mathbf{k})\times\mathbf{r} \tag{5.3}$$

can be disregarded. In this respect solution (5.2) is an analogue of solution (2.14) in the precession problem, which satisfies (2.9) also only in the limit when $(\boldsymbol{\Omega}\times\mathbf{k})\times\mathbf{r}$ can be neglected. Since the latter term and the term (5.3) have the same form, the analysis of the precession problem applies directly to the present problem, when $\eta^*(\mathbf{t}.\mathbf{k})\mathbf{t}$ is used in place of $\boldsymbol{\Omega}$. The same considerations are valid for a spheroidal shell with small ellipticity η instead of the spherical shell. Hence the results (3.19) and (4.12) describe the flow inside a spheroidal shell with a tidal bulge when $\boldsymbol{\Omega}$ is replaced by $\eta^*(\mathbf{t}.\mathbf{k})\mathbf{t}$.

† This problem has been recently attacked by Suess (1967).

6. A few remarks on the geophysical application

In their first paper Stewartson & Roberts (1963) discussed the relevance of the precession problem to the liquid core inside the rigid mantle of the precessing earth. They concluded that the viscosity does not alter the result of the inviscid theory by Poincaré (1910), who obtained a value of 10^{-5} for ϵ. Owing to the non-linear terms of the boundary-layer equation, the motion in the liquid core deviates from Poincaré's solution of constant vorticity even in the limit of vanishing viscosity. In the case of the earth's core, however, this effect is extremely small. The retrograde rotation of the order ϵ^2 fails by four orders of magnitude to account for the westward drift of the earth's magnetic field, if the drift represents the relative motion of the core.

Although the viscosity of the liquid core of the earth is not well known, the Ekman number for the core certainly is extremely small and probably is close to 10^{-15}. For this reason the stationary flow is likely to be unstable owing to the instability of the free shear layer, even though the amplitude ϵ^2 is very small. This suggests that the precession may be responsible for the wavelike processes which manifest themselves in the secular variation of the earth's magnetic field according to recent proposals by Hide (1966) and Malkus (1967).

The discussion of the effect of the tidal bulge on the motion of the earth's core becomes complicated because the elastic properties of the lower mantle are not well known and the direction $\mathbf{t}$ of the tidal bulge doesn't remain constant with respect to the inertial space. The latter difficulty can be removed if we neglect the fluctuating part. A rough estimate shows that the absolute value of the time average of the term (5.3) is of the order 10^{-8}. Thus the flow caused by the tidal bulge has both the same form and the same order of magnitude as the flow caused by the precession. The effects, however, compensate each other partly since the direction of the average of $\eta^*(\mathbf{k} \times \mathbf{t}) \times \mathbf{kt}.\mathbf{k}$ is opposite to $(\mathbf{k} \times \mathbf{\Omega}) \times \mathbf{k}$.

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REFERENCES

Greenspan, H. P. 1964 *J. Fluid Mech.* **20**, 673.
Greenspan, H. P. 1965 *J. Fluid Mech.* **22**, 449.
Greenspan, H. P. & Weinbaum, J. 1965 *J. Math. & Phys.* **44**, 66.
Hide, R. 1966 *Phil. Trans.* A **259**, 615.
Malkus, W. V. R. 1967 *Proc. Symp. Appl. Math.* **18**, 64.
Malkus, W. V. R. 1968 To appear.
Poincaré, H. 1910 *Bull. Astr.* **27**, 321.
Roberts, P. H. & Stewartson, K. 1963 *Astrophys. J.* **137**, 777.
Roberts, P. H. & Stewartson, K. 1965 *Proc. Camb. Phil. Soc.* **61**, 279.
Stewartson, K. & Roberts, P. H. 1963 *J. Fluid Mech.* **17**, 1.
Suess, S. T. 1967 Private communication.